

Partially Fractal Impedance Networks

D. Thompson

Sección Física, Pontificia Universidad Católica del Perú, Apdo. 1761, Lima 100, Peru

Z. Naturforsch. **53 a**, 819–822 (1998); received July 15, 1998

We analyse linear networks of impedances in the case when the impedances at every level differ by a factor of b from those at the previous level. Such networks can be used as models for rough surfaces, in which case there will exist a level of finest detail which must be taken into account in any calculation. We obtain an exact expression for the ratio of the impedance of the network to the outer impedance for an arbitrary number of elements in the network. We show that this class of networks shows a transition from a fractal geometric structure to a non-fractal structure according to the value of b . However, their effective impedance is never fractal.

Introduction

Lakhtakia *et al.* [1] have studied ladder circuits similar to that of Fig. 1 with a view to using them in the characterisation of rough surfaces. They give closed form results for semi-infinite networks, and recurrence relations for finite networks. Any attempt at modelling real surfaces must work with finite networks, as there always exists a smallest length scale present. In this paper we consider similar networks

of impedances, such as that shown in Fig. 1, where the impedances at each level differ from those at the previous level by a factor of b . In the second section we derive, in closed form, the effective impedance for this network for an arbitrary number of levels. We will show then that these networks show a transition between fractal and non-fractal properties in their geometric structures. We will also demonstrate that even when the structure is fractal, the effective impedance of the network is not fractal.

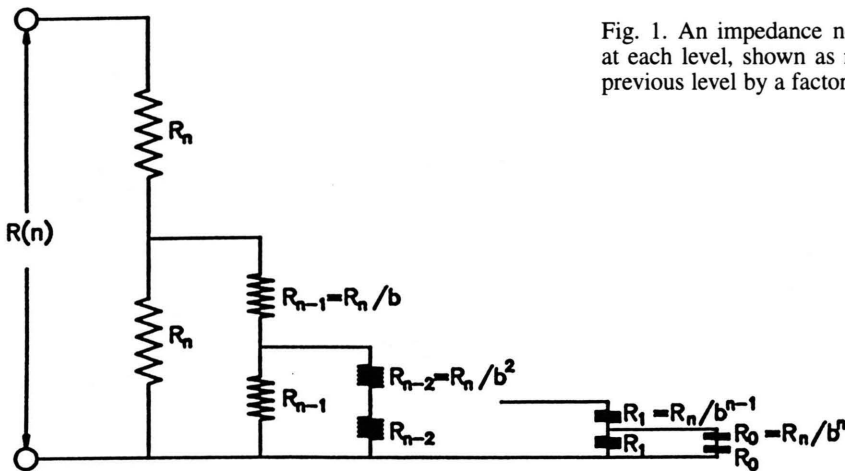


Fig. 1. An impedance network in which the impedances at each level, shown as resistors, differ from those at the previous level by a factor of b .

Reprint requests to Dr. David Thompson; Fax: +51 1 461 1785, E-mail: dthomps@fisica.pucp.edu.pe.

0932-0784 / 98 / 1000-0819 \$ 06.00 © Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

Impedance Calculations

We will use the following notation: Z_n is the value of the impedances at the n^{th} level; $Z(n)$ is the effective impedance of the network which contains n levels; and

$$x_n = Z(n)/Z_n \quad (1)$$

is the effective impedance measured in multiples of the impedances at the outer level. We will refer to the outer and inner levels of the network, the effective impedance being measured at the outer level. In Fig. 1 the outer impedances, $Z_n = R_n$, are the largest and the inner impedances, $Z_0 = R_0$, are the smallest, but it is not necessary that we should have $b > 1$. We will also need the initial value $Z(0) = 2Z_0$, which gives

$$x_0 = 2. \quad (2)$$

We should note that the circuit in Fig. 1 is only one of a whole family of similar circuits. In Fig. 1 a single branch of the impedances Z_{n-1} is placed in parallel with the impedance Z_n . We could connect N identical branches in parallel with the impedances at each level of the circuit.

The effective impedance $Z(n)$ with N branches in parallel at each level can obviously be calculated in the recursive manner

$$Z(n) = Z_n + \frac{1}{\frac{1}{Z_n} + \frac{N}{Z(n-1)}}, \quad (3)$$

which can be converted into a continued fraction by the successive substitution of the $Z(i)$,

$$Z(n) = Z_n + \frac{1}{\frac{1}{Z_n} + \frac{N}{Z_{n-1} + \frac{1}{\frac{1}{Z_{n-1}} + \frac{N}{Z_{n-2}}}}}. \quad (4)$$

Equation (3) can also be written as

$$\frac{Z(n)}{Z_n} = \frac{NZ_n/Z_{n-1} + 2Z(n-1)/Z_{n-1}}{NZ_n/Z_{n-1} + Z(n-1)/Z_{n-1}}, \quad (5)$$

which is a recurrence relation for x_n :

$$x_n = \frac{Nb + 2x_{n-1}}{Nb + x_{n-1}}. \quad (6)$$

We can view (6) as a non-linear mapping for x . That is, we can write $x_n = f(x_{n-1})$, where

$$f(x) = \frac{Nb + 2x}{Nb + x}. \quad (7)$$

The fixed points x_∞ of the mapping are those for which $x_\infty = f(x_\infty)$, that is the solutions of

$$x_\infty^2 + x_\infty(Nb - 2) - Nb = 0, \quad (8)$$

whose values are

$$x_\infty = 1 - (Nb/2) \pm \sqrt{(Nb/2)^2 + 1}. \quad (9)$$

In general, x_∞ is irrational. When $N = 1$ and $b = 1, 2$, and 4 , x_∞ takes on the interesting values of the golden ratio $\phi = (1 + \sqrt{5})/2$; $\sqrt{2}$; and $2/\phi$ [2].

These values of x_∞ are indeed the values of the effective impedance of the semi-infinite network measured in multiples of the outer impedance. A more physical way of seeing this is to note that when the network is semi-infinite, the addition of another outer level of impedances is equivalent to multiplying all of the impedances in the existing network by a factor of b . From the rules for combining impedances in series and in parallel one easily sees that multiplying all of the impedances in any network by a factor of b also multiplies the effective impedance by the same factor b . Replacing $Z(n-1)$ by $Z(n)/b$ in (3) leads directly to (8).

Although we have the effective impedance for a semi-infinite network, we should note that any application of these networks as models of rough surfaces should use a finite network because of the existence of a smallest length scale, such as the size of an atom or molecule, in a real surface. Obviously, the recursion relation given in (6), along with the initial value $x_0 = 2$, is sufficient to calculate x_n for any n .

However, we will now show that it is possible to solve (6) in closed form. Knowing that the fixed point of the mapping is the x_∞ given in (9), we introduce the new variable

$$y_n = x_n - x_\infty. \quad (10)$$

Using the condition for the fixed point, (8), the recursion relation for y_n can be written as

$$\frac{1}{y_n} = \frac{Nb + x_\infty}{2 - x_\infty} \frac{1}{y_{n-1}} + \frac{1}{2 - x_\infty}, \quad (11)$$

whose solution is

$$\frac{1}{y_n} = \left(\frac{Nb + x_\infty}{2 - x_\infty} \right)^n \frac{1}{y_0} + \frac{1}{2 - x_\infty} \sum_{m=0}^{n-1} \left(\frac{Nb + x_\infty}{2 - x_\infty} \right)^m. \quad (12)$$

The solution for x_n , using the value $x_0 = 2$ and the fixed point given by (9), is

$$x_n = 1 - \frac{Nb}{2} + \sqrt{1 + \left(\frac{Nb}{2} \right)^2 \frac{a_+ + a_-}{a_+ - a_-}}, \quad (13)$$

$$\text{where } a_\pm = \left(2 + Nb \pm \sqrt{(Nb)^2 + 4} \right)^{n+1}.$$

Fractal Dimensions

We now turn to the question of whether the impedance-networks that we are studying are fractal. One of the characteristics of a fractal is that, it is self-similar on all length scales, which means that, if one cuts a piece out of a fractal and then amplifies it to the size of the original, the amplified portion is indistinguishable from the original one. A more quantitative measure of whether a system is fractal is to determine how a quantity of interest, such as the content of the system, *e. g.*, the area or the volume, depends on the length scale used to describe the system. Let the measure of interest be $C(L)$ and let us suppose that it is proportional to a power d of the length scale L . That is to say, we suppose that we can write

$$C(L) \propto L^d. \quad (14)$$

For example, if the measure C is the volume of the system we would expect to find $d = 3$. A fractional value of d indicates a fractal system. The dimension d is obtained by measuring C at two different length scales $C(L_1)$ and $C(L_2)$ and calculating

$$d = \frac{\ln[C(L_1)/C(L_2)]}{\ln(L_1/L_2)}. \quad (15)$$

For the impedance networks under consideration we can look for two different dimensions, the structural dimension and the transport dimension. The first describes how the geometric content of the circuit scales with length, and the second describes how the effective impedance of the circuit scales. As a measure of the content of the system let us imagine that

the impedances are wires of equal cross section. The different impedances (resistances) are simply made of different lengths of wire. The measure C that we will use to describe the structure of the system is the total length of wire used. This is equivalent, for the general case of impedances, to calculating the sum of all of the impedances in the system. This is not, of course, the effective impedance of the system, because some of the impedances are connected in parallel.

We wish to compare the content of two networks, one with n levels of impedances and the other with $(n+1)$ levels, in both cases with the inner impedance Z_0 of the same size. The step from n to $n+1$ levels is a change of the length scale by a factor of b . The content $C(n)$ with n levels in the system is

$$C(n) = 2Z_n + \sum_{i=1}^n 2N^i \frac{Z_n}{b^i} = 2Z_n \frac{1 - (N/b)^{n+1}}{1 - (N/b)}. \quad (16)$$

Similarly we have

$$C(n+1) = 2Z_{n+1} \frac{1 - (N/b)^{n+2}}{1 - (N/b)} = 2bZ_n \frac{1 - (N/b)^{n+2}}{1 - (N/b)} \quad (17)$$

and

$$d = \frac{\ln \{ b[1 - (N/b)^{n+2}] / [1 - (N/b)^{n+1}] \}}{\ln(b)}. \quad (18)$$

The fractal dimension so obtained obviously depends on the value of n and it should, therefore, be evaluated in the limit of $n \rightarrow \infty$. The result then depends on the value of N/b . For $N/b \leq 1$ we have the uninteresting result $d = 1$. This corresponds to the fact that, in this case, the total amount of impedance present (the total length of wire) is finite even in the limit of having an infinite number of levels in the network, and that this quantity is proportional to the outer impedance. For $N/b > 1$ the dimension in the limit $n \rightarrow \infty$ is

$$d = \frac{\ln N}{\ln b}. \quad (19)$$

This is the fractal geometric dimension of the network.

We observe that there exists a critical value for N/b . When N/b is less than the critical value of one, the geometric structure is not fractal, but for larger values it is fractal. One can imagine constructing a network with a fixed value of N , and then varying the

value of b . As the value of b passes through N , the network will show a transition from fractal to non-fractal behaviour.

In the case of the transport dimension we compare the effective impedances of two networks, $Z(n) = Z_n x_n$ and $Z(n+1) = Z_{n+1} x_{n+1} = b Z_n x_{n+1}$, which gives

$$d_t = \frac{\ln(b x_{n+1} / x_n)}{\ln(b)}. \quad (20)$$

Given that the x_n tends to the limiting value x_∞ when $n \rightarrow \infty$, the limiting value for the transport dimension is

$$d_t = 1. \quad (21)$$

We see that even though the geometric structure can be fractal, the impedance is never fractal. However, there do exist impedance networks, such as the Sierpinski gasket [39], where both the geometric and the transport dimensions are fractal. It is interesting to note that when considering the transport properties of our networks the relevant parameter is Nb , but

when we consider the geometric properties it is N/b . This is a reflection of the fact that there does not exist a simple relation between geometric properties and transport properties for fractal structures.

Conclusions

Fractal impedance networks can be used to model the electrical properties of surfaces. One of the characteristics of a fractal is that it contains details on all length scales. We have looked at the question of what happens when there is a smallest length scale present in the problem, such as the atomic length scale that must always be present in the description of the surface. We have shown that it is possible to obtain a closed form expression for the impedance network shown in Fig. 1 for an arbitrary number of levels in the network.

The network is shown to have transition between fractal and non-fractal behaviour. It is a geometric fractal when $N/b > 1$ with fractal dimension $d = \ln N / \ln b$. However, even when the geometric structure is fractal, the impedance is not fractal.

- [1] A. Lakhtakia, R. Messier, V. V. Varadan, and V. K. Varadan, *Z. Naturforsch.* **43a**, 943 (1988).
- [2] D. Thompson, *Amer. J. Phys.* **65**, 88 (1997).

- [3] Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, *Phys. Rev. Lett.* **47**, 2311 (1987).